

Communication-Optimal Parallel Algorithm for Strassen's Matrix Multiplication

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The Plan

- I'll present a new parallel algorithm based on Strassen's matrix multiplication, called **C**ommunication **A**voiding **P**arallel **S**trassen
- The new Strassen-based parallel algorithm CAPS
 - is communication optimal
 - matches the lower bounds [B., Demmel, Holtz, Schwartz, '11]
 - is faster: in theory and in practice
- I'll also show performance results and talk about practical considerations for using Strassen and CAPS
- Strassen's algorithm is not just a theoretical idea: it can be practical in parallel and deserves further exploration

Outline

- 1 Motivation
- 2 Lower Bounds
- 3 Algorithms
- 4 Performance
- 5 Practical Considerations

Motivation: Strassen's fast matrix multiplication (1969)

Strassen's original algorithm uses 7 multiplies and 18 adds for $n = 2$.
Most importantly, it can be applied recursively.

$$Q_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$$

$$Q_2 = (A_{21} + A_{22}) \cdot B_{11}$$

$$Q_3 = A_{11} \cdot (B_{12} - B_{22})$$

$$Q_4 = A_{22} \cdot (B_{21} - B_{11})$$

$$Q_5 = (A_{11} + A_{12}) \cdot B_{22}$$

$$Q_6 = (A_{21} - A_{11}) \cdot (B_{11} + B_{12})$$

$$Q_7 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$$

$$\begin{matrix} n/2 \\ n/2 \end{matrix} \left\{ \begin{array}{|c|c|} \hline C_{11} & C_{12} \\ \hline C_{21} & C_{22} \\ \hline \end{array} \right\} = \begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \cdot \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array}$$

$$F(n) = 7 \cdot F(n/2) + O(n^2)$$

$$F(n) = \Theta\left(n^{\log_2 7}\right)$$

$$C_{11} = Q_1 + Q_4 - Q_5 + Q_7$$

$$C_{12} = Q_3 + Q_5$$

$$C_{21} = Q_2 + Q_4$$

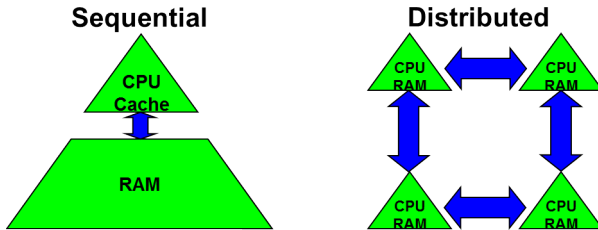
$$C_{22} = Q_1 - Q_2 + Q_3 + Q_6$$

$$\log_2 7 \approx 2.81$$

Motivation: communication costs

Two kinds of costs:

- Arithmetic (FLOPs)
- Communication: moving data
 - between levels of a memory hierarchy (sequential case)
 - over a network connecting processors (parallel case)
- Communication will only get more expensive relative to arithmetic



Motivation: communication costs

γ = time per FLOP

F = #Flops

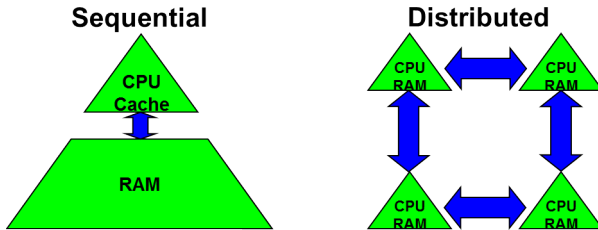
β = time per word

BW = #Words

α = time per message

L = #Messages

$$\text{Running time} = \gamma \cdot F + \beta \cdot BW + \alpha \cdot L$$



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Communication lower bounds for matrix multiplication



[Hong & Kung 81]

- Combinatorial proof
- Sequential only

Classical (cubic):

$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\log_2 8} M \right)$$



[Irony, Toledo, Tiskin 04]

- Geometric proof
- Sequential and parallel

$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\log_2 8} \frac{M}{P} \right)$$

n = matrix dimension, M = fast/local memory size, P = number of processors

Communication lower bounds for matrix multiplication

[B., Demmel, Holtz, Schwartz 11]:

- Sequential and parallel
- Graph expansion proof

Strassen:



$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\log_2 7} M \right)$$



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Strassen:

Strassen-like:

Classical (cubic):



$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\log_2 7} M \right)$$

$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\omega_0} M \right)$$

$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\log_2 8} M \right)$$



$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\log_2 7} \frac{M}{P} \right)$$

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Memory independent bound [B., Demmel, Holtz, Lipshitz, Schwartz 12]

Communication lower bounds for matrix multiplication

Algorithms attaining these bounds?

Strassen:



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Communication lower bounds for matrix multiplication

Algorithms attaining these bounds?

Strassen:



Sequential

$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\log_2 7} M \right)$$



Distributed

$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\log_2 7} \frac{M}{P} \right)$$



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**Our new
algorithm**

$$\Omega \left(\frac{n^2}{P^2 \log_2 7} \right)$$

[B., Demmel, Holtz,
Lipshitz, Schwartz 12]
[McColl & Tiskin 99]

Classical (cubic):

$$\Omega \left(\left(\frac{n}{\sqrt{M}} \right)^{\log_2 8} M \right)$$

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n = matrix dimension, M = fast/local memory size, P = number of processors

Lessons from lower bounds

① Don't use a classical algorithm for the communication

- Strassen can communicate less than classical

$$\text{Strassen: } \Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 7} \frac{M}{P}\right) \quad \text{Classical: } \Omega\left(\left(\frac{n}{\sqrt{M}}\right)^{\log_2 8} \frac{M}{P}\right)$$

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② Use all available memory

- Communication bound decreases with increased memory
- Up to a factor of $O(P^{1-2/\log_2 7})$ extra memory is useful

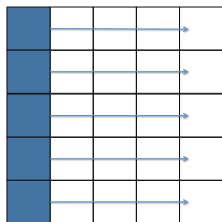
$$\text{Strassen: } \Omega\left(\max\left\{\left(\frac{n}{\sqrt{M}}\right)^{\log_2 7} \frac{M}{P}, \frac{n^2}{P^{2/\log_2 7}}\right\}\right)$$

Outline

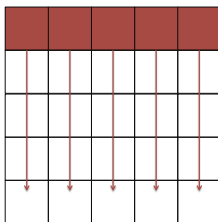
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Simple “2D” Classical Algorithm

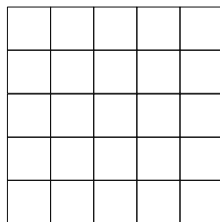
Here’s the basic communication pattern for the classical “2D” algorithm:



A



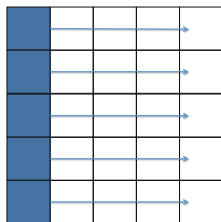
B



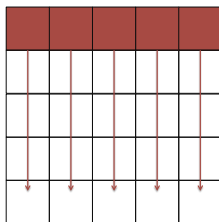
C

Simple “2D” Classical Algorithm

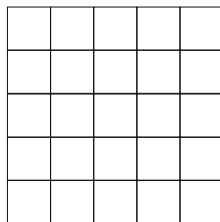
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A



B



C

- 2D: think Cannon or SUMMA
[Cannon 69, van de Geijn & Watts 97]
- 2.5D: think reduced communication by using more memory
[Solomonik & Demmel 11]

Previous parallel Strassen-based algorithms

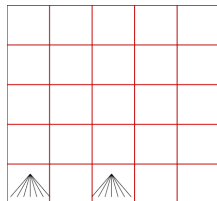
2D-Strassen: [Luo & Drake 95]

Run classical 2D inter-processors.

- Same communication costs as classical 2D.

Run Strassen locally.

- Can't use Strassen on the full matrix size.



Previous parallel Strassen-based algorithms

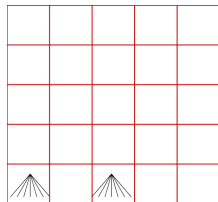
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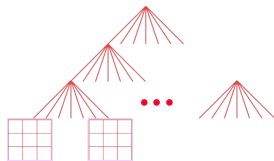
Strassen-2D: [Luo & Drake 95; Grayson, Shah, van de Geijn 95]

Run Strassen inter-processors

- This part can be done without communication.

Then run classical 2D.

- Communication costs grow exponentially with the number of Strassen steps.



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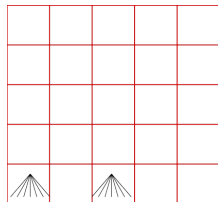
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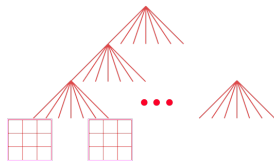
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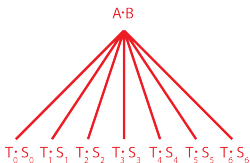


Neither is communication optimal, even if you use 2.5D

Main idea of CAPS algorithm

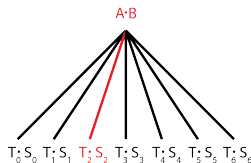
At each level of recursion tree, choose either breadth-first or depth-first traversal of the recursion tree

Breadth-First-Search (BFS)



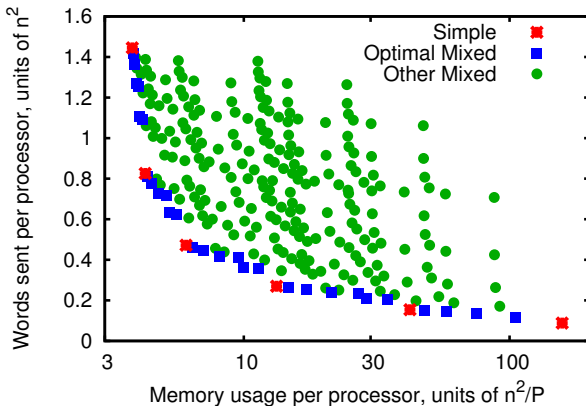
- Runs all 7 multiplies in parallel
 - each uses $P/7$ processors
- Requires 7/4 as much extra memory
- Requires communication, but
- All BFS minimizes communication if possible

Depth-First-Search (DFS)



- Runs all 7 multiplies sequentially
 - each uses all P processors
- Requires 1/4 as much extra memory
- No immediate communication
- Increases bandwidth by factor of 7/4
- Increases latency by factor of 7

Tuning the choices of BFS and DFS Steps



The memory and communication costs of all $\binom{10}{5} = 252$ possible interleavings of BFS and DFS steps for multiplying matrices of size $n = 351,232$ on $P = 7^5 = 16,807$ processors using 10 Strassen steps.

Asymptotic costs analysis

		Flops	Bandwidth Cost
Strassen	Lower Bound	$\frac{n^{\log_2 7}}{P}$	$\max \left\{ \frac{n^{\log_2 7}}{PM^{(\log_2 7)/2-1}}, \frac{n^2}{P^{2/\log_2 7}} \right\}$
	2D-Strassen	$\frac{n^{\log_2 7}}{P^{(\log_2 7-1)/2}}$	$\frac{n^2}{P^{1/2}}$
	Strassen-2D	$\left(\frac{7}{8}\right)^\ell \frac{n^3}{P}$	$\left(\frac{7}{4}\right)^\ell \frac{n^2}{P^{1/2}}$
	CAPS	$\frac{n^{\log_2 7}}{P}$	$\max \left\{ \frac{n^{\log_2 7}}{PM^{(\log_2 7)/2-1}}, \frac{n^2}{P^{2/\log_2 7}} \right\}$
Classical			

Asymptotic costs analysis

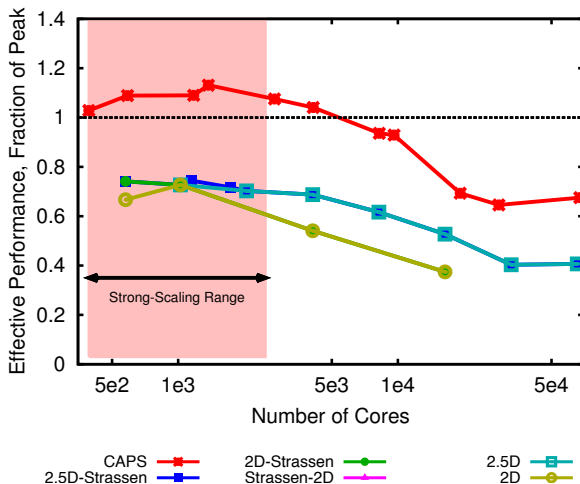
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	2D	$\frac{n^3}{P}$	$\frac{n^2}{P^{1/2}}$
	2.5D	$\frac{n^3}{P}$	$\max \left\{ \frac{n^3}{PM^{1/2}}, \frac{n^2}{P^{2/3}} \right\}$

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Performance of CAPS on large problems

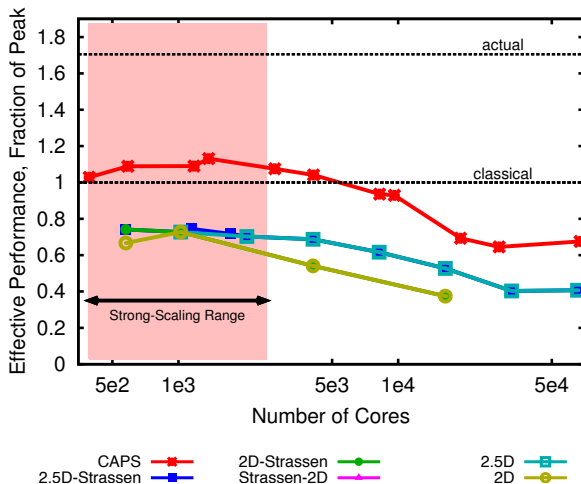
Strong-scaling on Intrepid (IBM BG/P), $n = 65,856$.



► Strassen-Winograd peak

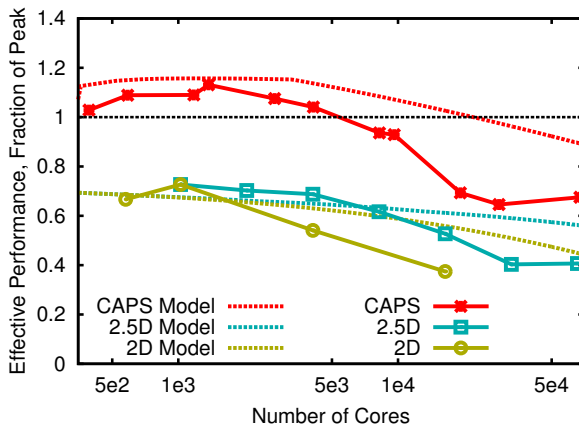
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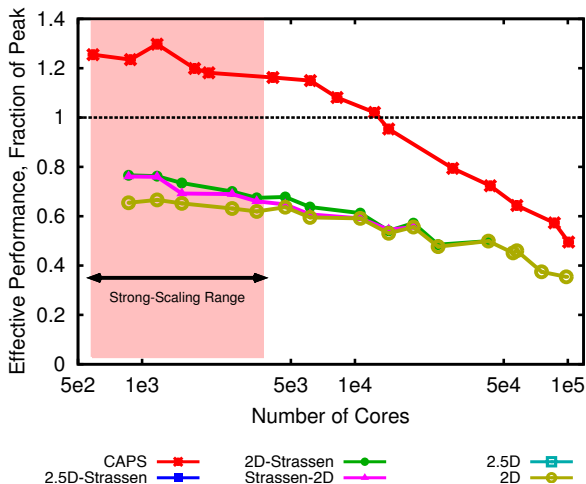
Performance: Model vs Actual



Comparison of the parallel models with the algorithms in strong scaling of matrix dimension $n = 65,856$ on Intrepid.

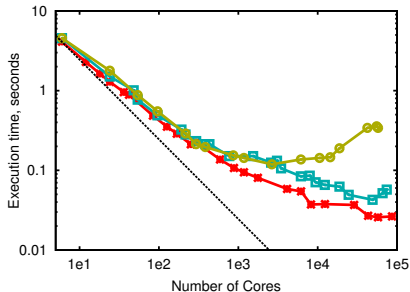
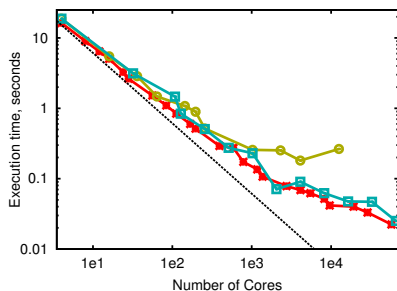
Performance of CAPS on large problems

Strong-scaling on Hopper (Cray XE6), $n = 131,712$.





Performance of CAPS on small (comm-bound) problems

Strong-scaling on Intrepid (left) and Hopper (right), $n = 4704$.



CAPS 
2.5D-Strassen 

2D-Strassen 
Strassen-2D 

2.5D 
2D 

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Practical Considerations for Strassen

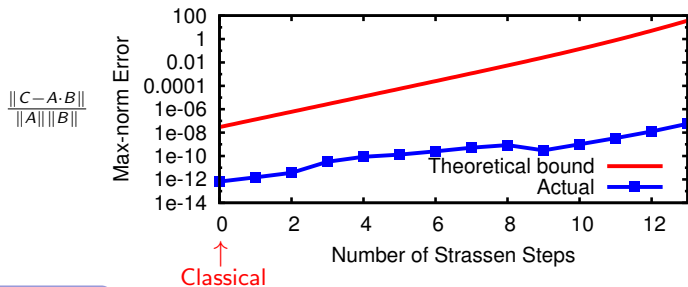
- ① Harder to reach actual peak performance
 - computation to communication ratio smaller than classical
- ② Additions and multiplications are no longer balanced
- ③ Architectures are based on powers of 2 not 7
 - CAPS prefers $P = m \cdot 7^k$
 - Intrepid requires allocation of power of two number of nodes
- ④ Stability bounds are not as strong as for classical

Stability - why you shouldn't worry

- CAPS has the same stability properties as any other Strassen (Strassen-Winograd) algorithm
- Weaker stability guarantee than classical, but still norm-wise stable
 - This can be improved with techniques like diagonal scaling

Stability - why you shouldn't worry

- CAPS has the same stability properties as any other Strassen (Strassen-Winograd) algorithm
- Weaker stability guarantee than classical, but still norm-wise stable
 - This can be improved with techniques like diagonal scaling
- Taking fewer Strassen steps improves the bound
- Theoretical bounds are pessimistic in the typical case



The CAPS matrix multiplication algorithm

- ① is communication optimal
- ② is faster: in theory and in practice
- ③ can be practical and should be used and improved

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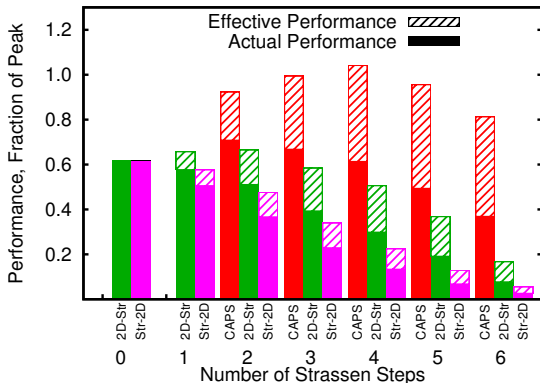
Thank You!

`www.eecs.berkeley.edu/~ballard`
`http://bebop.cs.berkeley.edu`

Extra slides

- 1 ▶ Performance: Model vs Actual
- 2 ▶ Time breakdown
- 3 ▶ DFS vs BFS
- 4 ▶ BFS on 7 Processors
- 5 ▶ Sequential Performance
- 6 ▶ Data Layout
- 7 ▶ Strassen-Winograd Algorithm
- 8 ▶ Actual vs Effective Performance
- 9 ▶ Small problem on Franklin
- 10 ▶ Big problem on Franklin
- 11 ▶ Diagonal Scaling
- 12 ▶ Open Problems

Effective vs Actual Performance

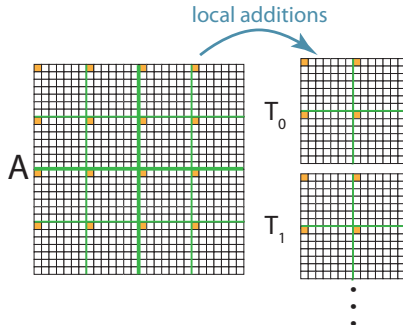
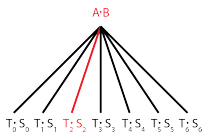


Efficiency at various numbers of Strassen steps, $n = 21952$, on 49 nodes (196 cores) of Intrepid.

Communication-Free DFS

Possible if each processor owns corresponding entries of four submatrices of A , B , and C . [Luo & Drake 95; Grayson, Shah, van de Geijn 95]

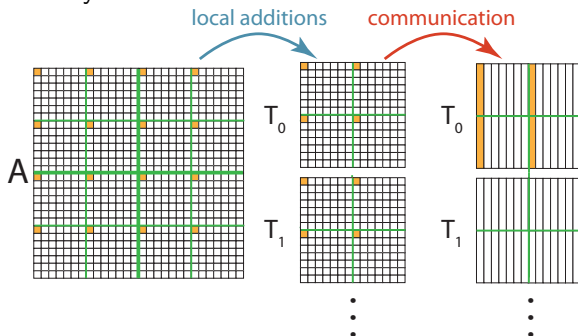
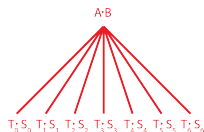
- Additions of submatrices of A to form the T_i (no communication)
- Additions of submatrices of B to form the S_i (no communication)
- Recursive calls $Q_i = T_i \cdot S_i$ (communication deeper in recursion tree)
- Additions of the Q_i to form submatrices of C (no communication)



Communication Pattern of BFS

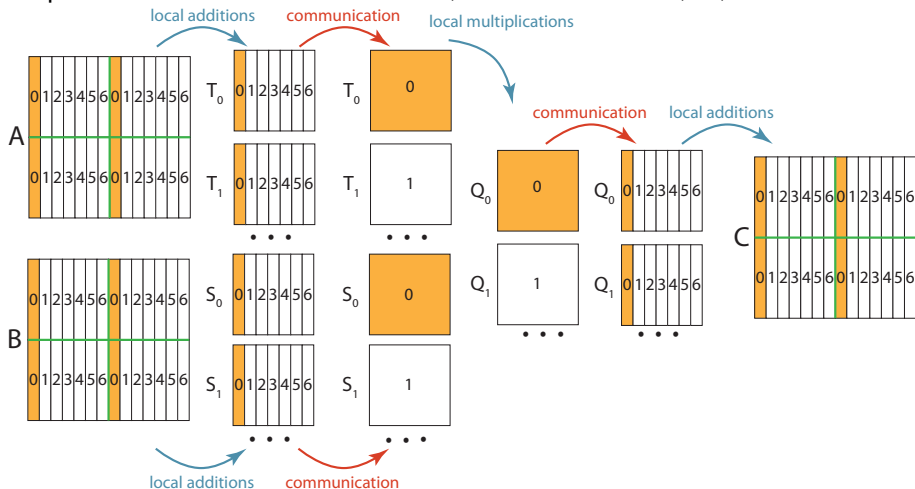
- Additions of submatrices of A, B to form T_i, S_i (no communication)
- Redistribution of the T_i, S_i (communication)
- Recursive calls $Q_i = T_i \cdot S_i$ (communication deeper in recursion tree)
- Redistribution of the Q_i (communication)
- Additions of the Q_i to form submatrices of C (no communication)

Redistributions are disjoint 7-way all-to-all communications.

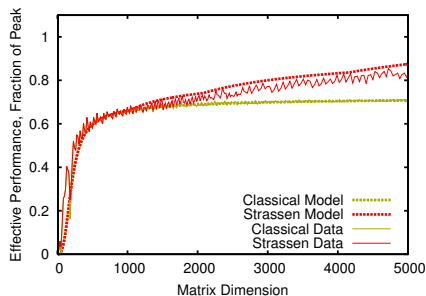


BFS on 7 Processors

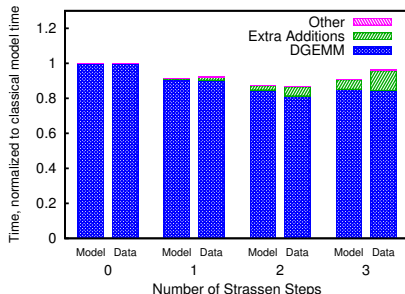
Requires 3 all-to-all communications, one for each of A , B , C



Sequential Performance

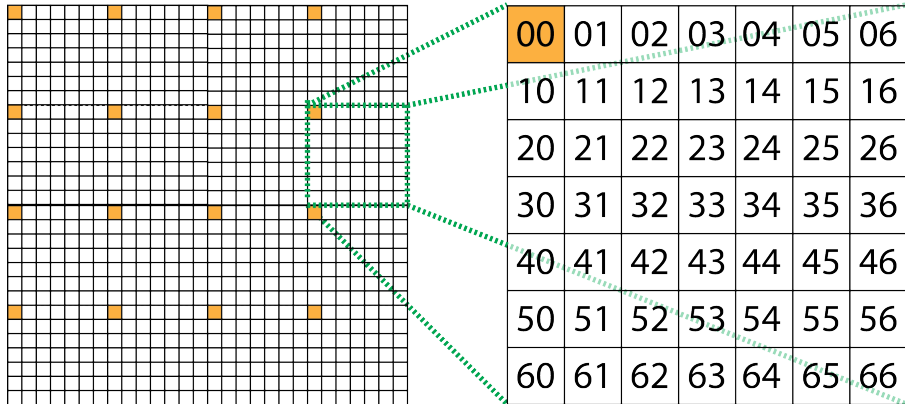


Comparison of the sequential model to the actual performance of classical and Strassen matrix multiplication on four cores (one node) of Intrepid.



Time breakdown comparison between the sequential model and the data for $n = 4097$. Both model and data times are normalized to the modeled classical algorithm time.

Data Layout

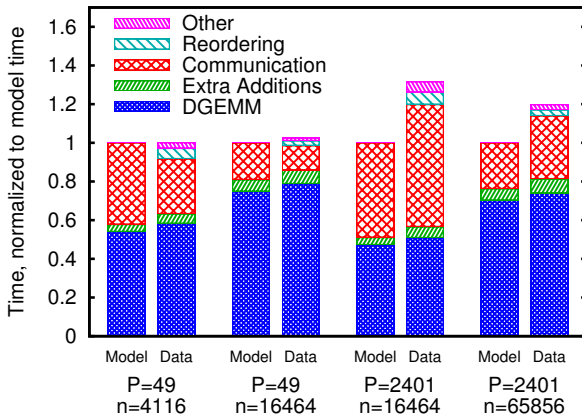


Strassen-Winograd Algorithm

$$\left(\begin{array}{c|c} C_{11} & C_{12} \\ \hline C_{21} & C_{22} \end{array} \right) = C = A \cdot B = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right) \cdot \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array} \right)$$

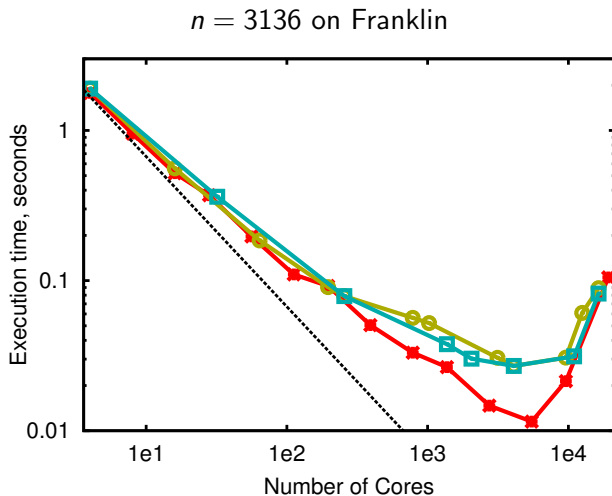
$S_0 = A_{11}$	$T_0 = B_{11}$	$Q_i = S_i \cdot T_i$
$S_1 = A_{12}$	$T_1 = B_{21}$	$U_1 = Q_i + Q_4$
$S_2 = A_{21} + A_{22}$	$T_2 = B_{12} + B_{11}$	$U_2 = U_1 + Q_5$
$S_3 = S_2 - A_{12}$	$T_3 = B_{22} - T_2$	$U_3 = U_1 + Q_5$
$S_4 = A_{11} - A_{21}$	$T_4 = B_{22} - B_{12}$	$C_{11} = Q_1 + Q_2$
$S_5 = A_{12} + S_3$	$T_5 = B_{22}$	$C_{12} = U_3 + Q_6$
$S_6 = A_{22}$	$T_6 = T_3 - B_{21}$	$C_{21} = U_2 - Q_7$
		$C_{22} = U_2 + Q_3$

Performance Breakdown: Model vs Actual



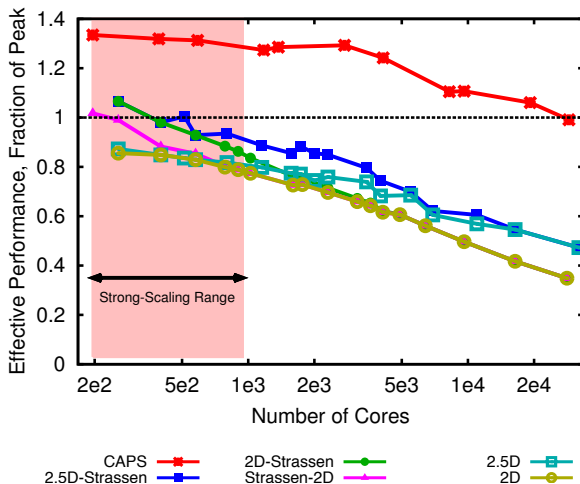
Time breakdown comparison between the parallel model and data on Intrepid. In each case the entire modeled execution time is normalized to 1.

Performance on Franklin for small problem



Performance of CAPS on large problem

Strong-scaling on Franklin (Cray XT4), $n = 94,080$.



Sequential recursive Strassen is communication optimal



- Run Strassen algorithm recursively.
- When blocks are small enough, work in local memory, so no further bandwidth cost

$$W(n, M) = \begin{cases} 7W(\frac{n}{2}, M) + O(n^2) & \text{if } 3n^2 > M \\ O(n^2) & \text{otherwise} \end{cases}$$

- Solution is

$$W(n, M) = O\left(\frac{n^{\omega_0}}{M^{\omega_0/2-1}}\right)$$

Diagonal Scaling

Outside scaling:

- Scale so each row of A and each column of B has unit norm.
- Explicitly:
 - Let $D_{ii}^A = (\|A(i, :)\|)^{-1}$, and $D_{jj}^B = (\|B(:, j)\|)^{-1}$.
 - Scale $A' = D^A A$, and $B' = B D^B$.
 - Use Strassen for the product $C' = A' B'$.
 - Unscale $C = (D^A)^{-1} C' (D^B)^{-1}$.

Diagonal Scaling

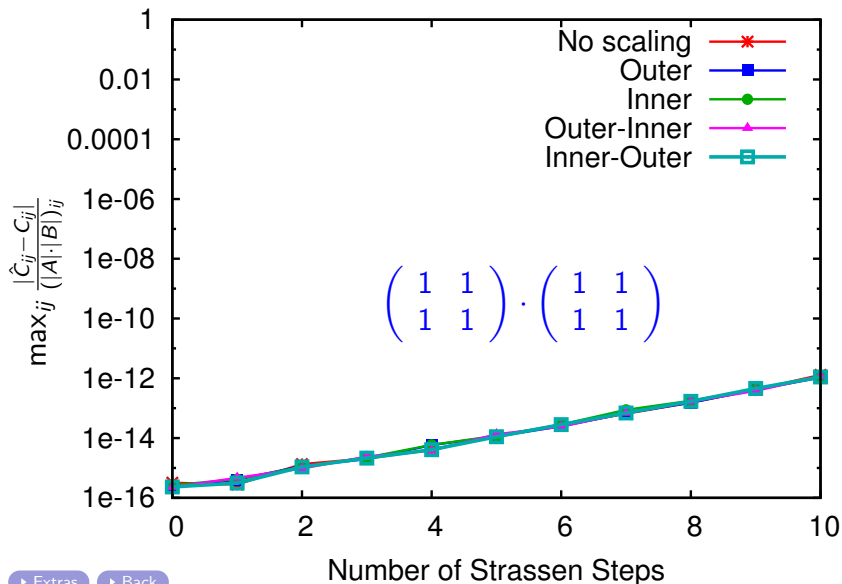
Outside scaling:

- Scale so each row of A and each column of B has unit norm.
- Explicitly:
 - Let $D_{ii}^A = (\|A(i, :)\|)^{-1}$, and $D_{jj}^B = (\|B(:, j)\|)^{-1}$.
 - Scale $A' = D^A A$, and $B' = B D^B$.
 - Use Strassen for the product $C' = A' B'$.
 - Unscale $C = (D^A)^{-1} C' (D^B)^{-1}$.

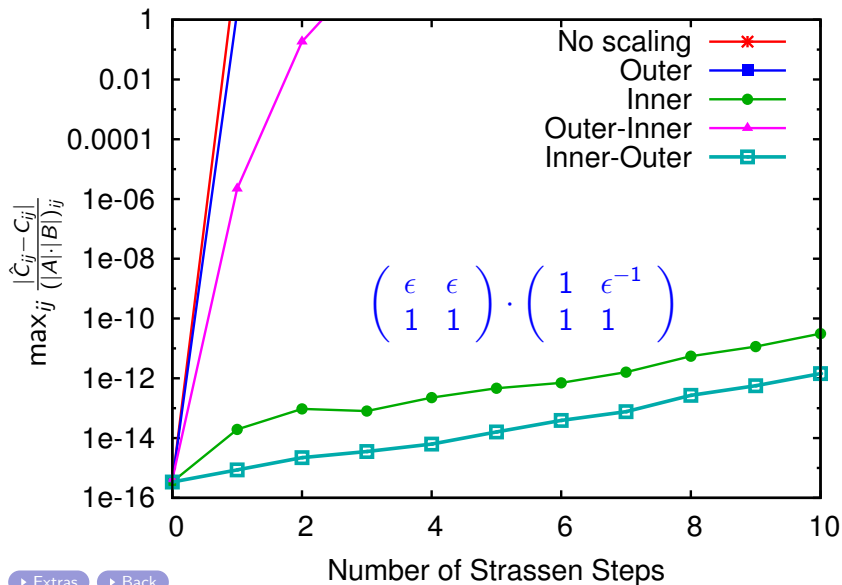
Inside scaling:

- Scale so each column of A has the same norm as the corresponding row of B .
- Explicitly:
 - Let $D_{ii} = (\|A(:, i)\| / \|B(i, :)\|)^{-1/2}$.
 - Scale $A' = A D$, and $B' = D^{-1} B$.
 - Use Strassen for the product $C = A' B'$.

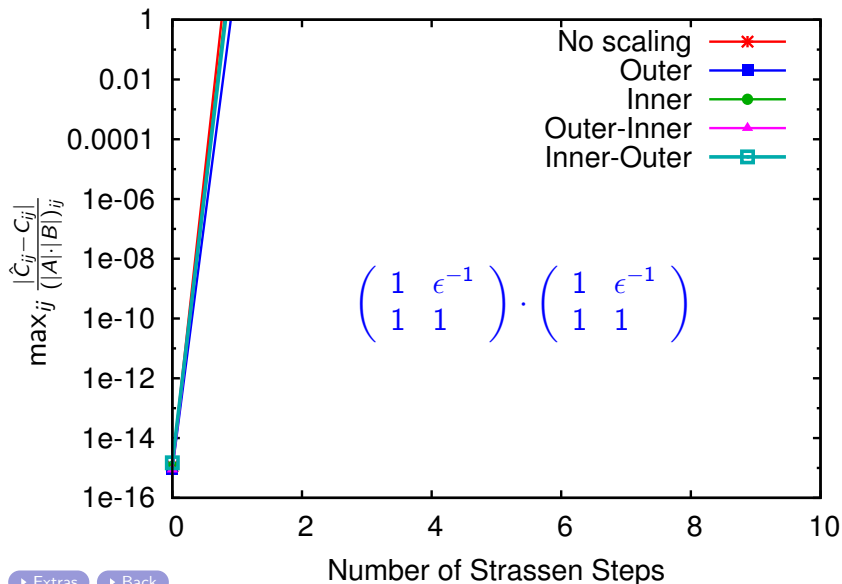
Stability: easy case



Stability: more interesting case



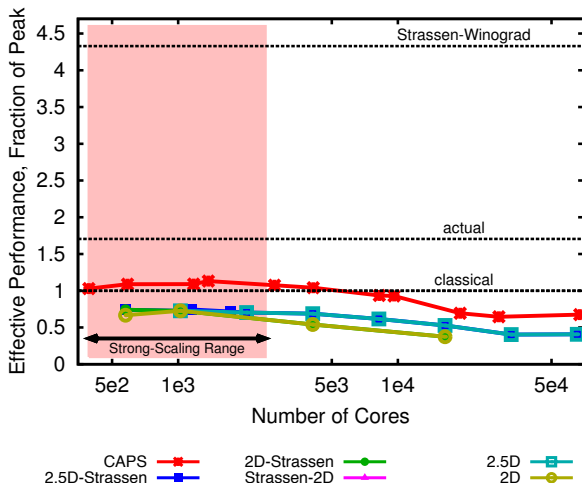
Stability: problems scaling can't fix



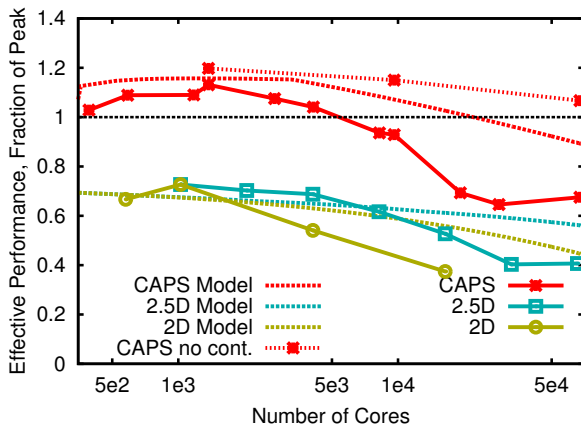
- Our parallelization approach extends to other matrix multiplication algorithms:
 - classical matrix multiplication (matching the 2.5D algorithm)
 - other fast matrix multiplication algorithms
- And to other algorithms with recursive formulations?
- Make use of CAPS within other linear algebra algorithms

Performance of CAPS on large problems

Strong-scaling on Intrepid (IBM BG/P), $n = 65,856$.



Performance: Model vs Actual



Comparison of the parallel models with the algorithms in strong scaling of matrix dimension $n = 65,856$ on Intrepid.

Extra slides

- 1 ▶ Performance: Model vs Actual
- 2 ▶ Time breakdown
- 3 ▶ DFS vs BFS
- 4 ▶ BFS on 7 Processors
- 5 ▶ Sequential Performance
- 6 ▶ Data Layout
- 7 ▶ Strassen-Winograd Algorithm
- 8 ▶ Actual vs Effective Performance
- 9 ▶ Small problem on Franklin
- 10 ▶ Big problem on Franklin
- 11 ▶ Diagonal Scaling
- 12 ▶ Open Problems